

Regular Solutions of Second-Order Stationary Hamilton–Jacobi Equations

FAUSTO GOZZI

Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti n. 2, 56127 Pisa, Italy

AND

ELISABETH ROUY

Institut Elie Cartan, INRIA Lorraine, CNRS UMR 9973, B.P. 239,

View metadata, citation and similar papers at core.ac.uk

Received June 12, 1995

We study a second order stationary Hamilton–Jacobi equation in infinite dimension. This equation is nonlinear and convex with respect to the first-order term. We use properties of the transition semigroup associated to the linear equation to write the Hamilton–Jacobi equation in integral form and we prove existence, uniqueness and regularity of a solution by the theory of maximal monotone operators. We also prove that this solution is the pointwise limit of a uniformly bounded sequence of classical solutions of approximating problems. Finally, the solution is the value function of the associated optimal stochastic control problem. Some examples are given. © 1996 Academic Press, Inc.

1. INTRODUCTION

We study the following second-order Hamilton–Jacobi equation:

$$\lambda u(x) = \frac{1}{2} \operatorname{Tr}[Qu_{xx}(x)] + \langle Ax + F(x), u_x(x) \rangle - H(u_x(x)) + \psi(x), \quad x \in X \quad (1.1)$$

where X is a separable Hilbert space, A is the infinitesimal generator of a strongly continuous semigroup of negative type on X , Q is a nonnegative self-adjoint operator on X which is not necessarily nuclear, F is a bounded Lipschitz continuous function which takes its values in X and finally H is a Lipschitz continuous real valued function.

For all $\lambda > 0$ and all ψ uniformly continuous and bounded real valued function, we define the mild solution of (1.1) as the solution of the integral equation

$$u(x) = \int_0^{+\infty} e^{-\lambda t} P_t[\langle F(\cdot), u_x \rangle - H(u_x) + \psi](x) dt \quad (1.2)$$

where $\{P_t; t \geq 0\}$ is the Ornstein–Uhlenbeck semigroup associated to the parabolic equation

$$\frac{\partial v}{\partial t} = \frac{1}{2} \text{Tr}[Qv_{xx}] + \langle Ax, v_x \rangle$$

$$v(0) = \varphi.$$

The properties of the semigroup $\{P_t; t \geq 0\}$, when it acts on the space of uniformly continuous and bounded functions on X , have been studied, for instance, in [16], [6], [7] and [18]. By a fixed point argument, we first prove, under the so-called null controllability assumption which links Q and A , existence and uniqueness of a solution of (1.2) in the Banach space of uniformly continuous Fréchet differentiable functions for large enough $\lambda > 0$. Besides, u is the resolvent of a unique accretive operator whose resolvent set contains $(0, +\infty)$. This implies that (1.1) has a mild solution for all $\lambda > 0$ and all ψ . Moreover, if ψ is Fréchet differentiable, u is twice differentiable.

Following [7] and [18], we prove that the mild solution of (1.1) is the pointwise limit of a uniformly bounded sequence of classical solutions of equations of type (1.1) (i.e. with a different “ ψ ”) which approximate (1.1) in a suitable sense.

Finally, when H is given by

$$H(p) = \sup_{|z| \leq R} \{\langle z, p \rangle - h(z)\}, \quad (1.3)$$

where h is a convex l.s.c. function on $\{|z| \leq R\}$ for some $R > 0$, then the mild solution of (1.1) is the value function of the following optimal stochastic control problem: the dynamic is the mild solution of the stochastic differential equation

$$dy(s) = (Ay(s) + F(y(s)) - z(s)) ds + \sqrt{Q} dW(s), \quad s > 0$$

$$y(0) = x$$

where the control z lies in the space $M_{\mathcal{W}}^2(0, +\infty; X)$ of all stochastic processes which are square integrable and adapted to the white noise W ; and the value function is the minimal cost defined by

$$V(x) = \inf_{\substack{z \in M_{\mathcal{W}}^2(0, +\infty; X) \\ |z| \leq R}} \left\{ \int_0^{+\infty} e^{-\lambda s} [h(z(s)) + \psi(y(s))] ds \right\}.$$

The proof is based on Itô's formula. When H is sufficiently smooth, there exist an optimal control z^* and an optimal trajectory y^* i.e. processes which satisfy

$$V(x) = \int_0^{+\infty} e^{-\lambda s} [h(z^*(s)) + \psi(y^*(s))] ds.$$

The optimal control is given by the feedback formula

$$z^*(s) = H_x(u_x(y^*(s)))$$

and the optimal trajectory is the mild solution of the closed loop equation:

$$\begin{aligned} dy(s) &= (Ay(s) + F(y(s)) - H_x(u_x(y^*(s)))) ds + \sqrt{Q} dW(s), \quad s > 0 \\ y(0) &= x. \end{aligned}$$

We observe that, as in [4], [18] and in [19], our assumptions cover the case when A is the Laplace operator in a bounded domain in \mathbb{R}^N ($N \leq 3$) with Dirichlet or Neumann boundary conditions. If $N=1$ we can take $Q=I$, while for $N=2, 3$ we have to deal with an appropriate compact operator Q . Similarly (see §6) we can cover the case when A is the bi-Laplacian in dimension $N \leq 7$ ($N \leq 3$ if we take $Q=I$). Moreover, in the finite dimensional case, our results state existence and uniqueness of regular solutions in the uniformly elliptic case (see Remark 2.6(iii)).

Several results on second order Hamilton–Jacobi equations are obtained by the approach of viscosity solutions. For a presentation of the argument in the finite dimensional case see [10], [17] and the references quoted therein. For the infinite dimensional case, see [23] and [28]. In particular, in [28] the author states existence and uniqueness of viscosity solutions (which are a priori nondifferentiable) for a wide class of second order partial differential equations. When Q is nuclear equation (1.1) falls into this class.

Other papers concerning more regular solutions of second order Hamilton–Jacobi equations in infinite dimensions are [2], [20], [13], [4], [3], [18] and [19] for the evolution case and [9] for the stationary case. In particular the last paper studies (1.1) in the space of functions that

are square integrable on X with respect to the invariant measure of the Ornstein–Uhlenbeck process (see [16] for the properties of such measure).

The plan of the paper is the following. Section 2 is dedicated to notations and preliminary results on the linear case. In Section 3 we prove existence and uniqueness of the mild solution of (1.1). The purpose of Section 4 is to prove that mild solutions are limit of classical solutions. In Section 5 we prove that the mild solution of (1.1) is the value function of the optimal control problem defined above as soon as (1.3) holds. Finally, in Section 6 we give two examples.

2. PRELIMINARIES

2.1. Notations

Let X and Y be two separable Hilbert spaces endowed with the scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_Y$ and the norms $|\cdot|$ and $|\cdot|_Y$.

We denote by $\mathbf{C}_b(X, Y)$ the Banach space of all functions $\varphi: X \rightarrow Y$ which are uniformly continuous and bounded on X and by $\|\cdot\|$ the usual norm on $\mathbf{C}_b(X, Y)$ defined by

$$\|\varphi\| = \sup_{x \in X} |\varphi(x)|_Y.$$

For $k \in \mathbb{N}$, we denote by $\mathbf{C}_b^k(X, Y)$ the set of all functions of $\mathbf{C}_b(X, Y)$ whose all Fréchet derivatives up to the order k are uniformly continuous and bounded on X . It is a Banach space when endowed with the norm

$$\|\varphi\|_k = \sum_{h=0}^k \sup_{x \in H} |D^h \varphi(x)|_Y.$$

We also denote by $\mathbf{C}^{0,1}(X, Y)$ the space of all Lipschitz continuous functions from X to Y ; we define the semi-norm

$$\|\varphi\|_{0,1} = \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|}; x, y \in X; x \neq y \right\},$$

and we set

$$\mathbf{C}^{1,1}(X, Y) = \{ \varphi \in \mathbf{C}^{0,1}(X, Y) \text{ Fréchet-differentiable s.t. } \|D\varphi\|_{0,1} < \infty \}.$$

If $Y = \mathbb{R}$ then we shall write $\mathbf{C}_b(X)$, $\mathbf{C}_b^k(X)$ and $\mathbf{C}^{k,1}(X)$ instead of $\mathbf{C}_b(X, \mathbb{R})$, $\mathbf{C}_b^k(X, \mathbb{R})$ and $\mathbf{C}^{k,1}(X, \mathbb{R})$.

Finally, we introduce the space $\mathbf{C}_b^\alpha(X)$ for $\alpha \in (0, 1)$ which is the Banach space of α -Hölder continuous and bounded functions. Similarly, $\mathbf{C}_b^{1+\alpha}(X)$ is

the space of all uniformly continuous bounded and Fréchet-differentiable functions with α -Hölder continuous bounded derivative. We denote by $\|\varphi\|_\alpha$ the Hölderian norm of $\varphi \in C_b^\alpha(X)$.

2.2. The Linear Problem

From now on, we shall assume the following

HYPOTHESIS 2.1. (i) A is the infinitesimal generator of a strongly continuous semigroup of negative type e^{tA} on X . For simplicity we also assume that $\|e^{tA}\| \leq 1$ for every $t \geq 0$.

(ii) Q is a bounded self-adjoint nonnegative operator on X .

(iii) W is a cylindrical Wiener process which takes its values in X and is defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Let then Q_t be, for all $t \geq 0$, the operator defined by

$$Q_t = \int_0^t e^{sA} Q e^{sA*} ds$$

and assume

HYPOTHESIS 2.2.

$$\text{Tr } Q_t < +\infty, \quad \forall t > 0 \quad (2.1)$$

$$\text{Im } e^{tA} \subset \text{Im } Q_t^{1/2}, \quad \forall t > 0. \quad (2.2)$$

As it is described in [16] Ch. 9 and in [7] (see also [15]), these assumptions guarantee that the transition semigroup $(P_t)_{t \geq 0}$ defined on $C_b(X)$ by $P_t \varphi(x) = \mathbb{E}[\varphi(Z(t, x))]$ for all $t \geq 0$, where $Z(t, x)$ is given by

$$Z(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} \sqrt{Q} dW(s),$$

is infinitely Fréchet-differentiable and is the solution in a mild sense of the Kolmogorov equation

$$\begin{aligned} v_t &= \frac{1}{2} \text{Tr}[Q v_{xx}] + \langle Ax, v_x \rangle, \quad t > 0 \\ v(0, \cdot) &= \varphi. \end{aligned} \quad (2.3)$$

Remark 2.3. (i) From (2.2) and the closed graph theorem it follows that for every $t > 0$, the operator $\Gamma(t) = Q_t^{-1/2} e^{tA}$ is well defined and bounded. This yields, together with (2.1) that for every $t > 0$ the operator $e^{tA} = Q_t^{1/2} \Gamma(t)$ is Hilbert-Schmidt on X .

(ii) We recall (see [16]) that (2.2) is equivalent to the null controllability of the deterministic system

$$\xi'(t) = A\xi(t) + \sqrt{Q} z(t), \quad \xi(0) = \xi_0.$$

Note that the null controllability assumption is crucial to guarantee the regularity, with respect to x of the solution of (2.3). In the finite dimensional case it reduces to the Hörmander hypoellipticity condition

$$Q_t > 0, \quad \forall t > 0$$

(see [21], [14], [26]).

We have the following result (see e.g. [16] or [18]).

PROPOSITION 2.4. *Assume that Hypotheses 2.1 and 2.2 hold true. Then, if $\varphi \in \mathbf{C}_b(X)$, $P_t \varphi \in \mathbf{C}_b^\infty(X)$ and we have the following estimates*

$$\begin{aligned} \|P_t \varphi\| &\leq \|\varphi\| \\ \|D_x P_t \varphi\| &\leq \|\Gamma(t)\| \|\varphi\| \quad , \quad \forall t > 0, \quad \forall \varphi \in \mathbf{C}_b(X), \\ \|D_{xx} P_t \varphi\| &\leq C \|\Gamma(t)\|^2 \|\varphi\| \\ \|D_x P_t \varphi\| &\leq \|\varphi_x\| \\ \|D_{xx} P_t \varphi\| &\leq \|\Gamma(t)\| \|\varphi_x\| \quad , \quad \forall t > 0, \quad \forall \varphi \in \mathbf{C}_b^1(X) \end{aligned}$$

for some positive constant C .

We shall also make the following assumption on $\Gamma(t)$:

HYPOTHESIS 2.5.

$\exists \varepsilon_0 > 0$ such that $\|\Gamma(t)\|^{1+\varepsilon_0}$ is locally integrable at 0.

Remark 2.6. (i) It can be proven that $t \mapsto \|\Gamma(t)\|$ is non increasing and, consequently, that $\|\Gamma(t)\|$ is bounded for t large enough and Hypothesis 2.5 implies

$$\begin{aligned} \alpha(\lambda) &= \int_0^{+\infty} e^{-\lambda t} \|\Gamma(t)\| dt < +\infty \\ \int_0^{+\infty} e^{-\lambda t} \|\Gamma(t)\|^{1+\varepsilon_0} dt &< +\infty \end{aligned} \tag{2.4}$$

for all $\lambda > 0$ and

$$\alpha(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty. \tag{2.5}$$

(ii) Hypothesis 2.5 is satisfied if and only if there exist $C > 0$, $t_0 > 0$ and $\beta \in (0, 1)$ such that

$$\|F(t)\| \leq \frac{C}{t^\beta}, \quad \text{for } 0 < t < t_0.$$

(iii) When the dimension of X is finite, then Hypothesis 2.5 reduces to assume that the operator Q is invertible i.e. the uniform ellipticity condition (see [27]).

3. RESOLUTION OF THE HAMILTON-JACOBI EQUATION

In this section, we are considering general nonlinear second-order Hamilton-Jacobi equations of the form

$$\lambda u = \frac{1}{2} \text{Tr}[Qu_{xx}] + \langle Ax + F(x), u_x \rangle - H(u_x) + \psi, \quad x \in X \quad (3.1)$$

under the Hypotheses 2.1, 2.2, 2.5 and

HYPOTHESIS 3.1 (on H and F). (i) $H: X \rightarrow \mathbb{R}$ is a Lipschitz continuous convex function, of Lipschitz constant K

(ii) $F: X \mapsto X$ is a Lipschitz continuous and bounded function of Lipschitz constant L .

We want to solve this equation for any $\lambda > 0$ and any $\psi \in \mathbf{C}_b(X)$.

We are going to define a class of solutions in $\mathbf{C}_b^1(X)$ for which we can prove existence and uniqueness for any $\lambda > 0$ and any $\psi \in \mathbf{C}_b(X)$. This will also imply that the solutions of (3.1) are the resolvents of a nonlinear m -dissipative operator defined on a domain included in $\mathbf{C}_b^1(X)$. We shall then see in the next sections that the notion of solutions introduced here is coherent with both the notion of classical solutions and value functions of optimal stochastic control problems.

3.1. Mild Solutions

We define here a notion of mild solutions which satisfy an integral equation.

DEFINITION 3.2. Let λ be a positive constant and $\psi \in \mathbf{C}_b(X)$. We shall say that $u \in \mathbf{C}_b^1(X)$ is a mild solution of (3.1) if and only if u satisfies the following integral equation for all $x \in X$:

$$u(x) = \int_0^{+\infty} e^{-\lambda t} P_t[\psi + \langle F, u_x \rangle - H(u_x)](x) dt. \quad (3.2)$$

We shall then denote $F_\lambda(\psi) = \{u \in \mathbf{C}_b^1(X) \text{ s.t. (3.2) holds}\}.$

Then we have the following theorem which gives existence, uniqueness and regularity of mild solutions of (3.1) for $\lambda > 0$ large enough.

THEOREM 3.3. *Assume that Hypotheses 2.1, 2.2, 2.5 and 3.1 hold. Then there exists a $\lambda_0 > 0$ such that:*

- (i) *for all $\lambda \geq \lambda_0$ and for all $\psi \in \mathbf{C}_b(X)$, (3.1) has a unique mild solution $F_\lambda(\psi)$ in $\mathbf{C}_b^1(X)$*
- (ii) *for all $\lambda \geq \lambda_0$, if $\psi \in \mathbf{C}_b^1(X)$ then $F_\lambda(\psi) \in \mathbf{C}_b^2(X)$*
- (iii) *for all $\lambda \geq \lambda_0$ and all $\varphi, \psi \in \mathbf{C}_b(X)$, we have*

$$\|F_\lambda(\varphi) - F_\lambda(\psi)\| \leq \frac{1}{\lambda} \|\varphi - \psi\| \quad (3.3)$$

- (iv) *for all $\lambda, \mu \geq \lambda_0$, for all $\psi \in \mathbf{C}_b(X)$, the following so-called identity of the resolvents holds:*

$$F_\lambda(\psi) = F_\mu(\psi + (\mu - \lambda) F_\lambda(\psi)). \quad (3.4)$$

In order to prove this theorem, we shall introduce some notations. For all $\lambda > 0$, we call T_λ the linear operator defined on $\mathbf{C}_b(X)$ by

$$T_\lambda \psi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \psi(x) dt, \quad \forall x \in X$$

and T_λ^ψ the nonlinear operator defined on $\mathbf{C}_b^1(X)$ by

$$T_\lambda^\psi u(x) = T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)](x), \quad \forall x \in X \text{ and } \psi \in \mathbf{C}_b(X).$$

Let us remark that, since $(P_t)_{t \geq 0}$ is a contraction semigroup, $T_\lambda \psi(x)$ and therefore $T_\lambda^\psi u(x)$ are well defined. Moreover, (3.2) is equivalent to $u(x) = T_\lambda^\psi u(x)$, so that $u \in \mathbf{C}_b^1(X)$ is a mild solution of (3.1) if and only if it is a fixed point of T_λ^ψ .

PROPOSITION 3.4. *For all $\lambda > 0$ and for all $\psi \in \mathbf{C}_b(X)$, the following statements hold:*

- (i) $T_\lambda \psi \in \mathbf{C}_b^1(X)$ and

$$\|T_\lambda \psi\| \leq \frac{1}{\lambda} \|\psi\|$$

$$\|(T_\lambda \psi)_x\| \leq \alpha(\lambda) \|\psi\|$$

(ii) if $\psi \in \mathbf{C}_b^1(X)$, then $T_\lambda \psi \in \mathbf{C}_b^2(X)$ and

$$\|(T_\lambda \psi)_x\| \leq \frac{1}{\lambda} \|\psi_x\|$$

$$\|(T_\lambda \psi)_{xx}\| \leq \alpha(\lambda) \|\psi_x\|.$$

Proof. It is obvious by using Proposition 2.4 and (2.4). ■

PROPOSITION 3.5. For all $\lambda > 0$, we have the following statements:

- (i) if $\psi \in \mathbf{C}_b(X)$ then T_λ^ψ maps $\mathbf{C}_b^1(X)$ into itself.
- (ii) for all $u, v \in \mathbf{C}_b^1(X)$,

$$\|T_\lambda^\psi u - T_\lambda^\psi v\|_1 \leq \left(\frac{1}{\lambda} + \alpha(\lambda)\right) (\|F\| + K) \|u_x - v_x\|,$$

and thus there exists a $\lambda_0 > 0$ such that, for all $\lambda \geq \lambda_0$ and for all $\psi \in \mathbf{C}_b(X)$, T_λ^ψ is a contraction in $\mathbf{C}_b^1(X)$.

Proof. The first point is a direct consequence of Proposition 3.4. The second one relies on the fact that $\alpha(\lambda)$ is a non-increasing function and that (2.5) holds. ■

Proof of Theorem 3.3(i). This last proposition proves the first item of our theorem since for all $\lambda \geq \lambda_0$ and for all $\psi \in \mathbf{C}_b(X)$, T_λ^ψ has a unique fixed point and thus (3.1) a unique mild solution. ■

In order to prove the second point, we shall proceed with interpolation and bootstrap methods.

Recall that $\varepsilon_0 > 0$ is a parameter defined in Hypothesis 2.5 such that the Laplace transform of $t \mapsto \|F(t)\|^{1+\varepsilon_0}$ is well defined (see Remark 2.6(i)).

LEMMA 3.6. For any $\lambda > 0$, T_λ has the following regularizing effects:

- (i) If $\varphi \in \mathbf{C}_b(X)$ then $T_\lambda \varphi \in \mathbf{C}_b^{1+\varepsilon_0}(X)$.
- (ii) If $\varphi \in \mathbf{C}_b^\alpha(X)$ for some $\alpha \in [0, 1)$ then $T_\lambda \varphi \in \mathbf{C}_b^{1+\varepsilon_0+\alpha-\varepsilon_0\alpha}(X)$.
- (iii) If $\varphi \in \mathbf{C}_b^{1-\varepsilon_0}(X)$ then $T_\lambda \varphi \in \mathbf{C}_b^2(X)$.

Proof. By following, for instance, [24] (see also the references quoted therein), we can deduce, from the following estimates

$$\begin{aligned} \|(P_t \varphi)_x\|_1 &\leq C_1 \|F(t)\|^2 \|\varphi\| \\ \|(P_t \varphi)_x\| &\leq \|F(t)\| \|\varphi\| \end{aligned}, \quad \forall \varphi \in \mathbf{C}_b(X), \quad \text{for } 0 < t < t_0$$

for some positive constant C_1 and some $t_0 > 0$, that for all $\theta \in (0, 1)$ and for all $\varphi \in \mathbf{C}_b(X)$,

$$\|(P_t \varphi)_x\|_\theta \leq C_\theta \|\Gamma(t)\|^{1+\theta} \|\varphi\|, \quad \text{for } 0 < t < t_0,$$

for some positive constant C_θ , since the interpolation space between $\mathbf{C}_b(X)$ and $\mathbf{C}_b^1(X)$ is $\mathbf{C}_b^\gamma(X)$ (see [5]). By choosing $\theta = \varepsilon_0$, we get the first item of our lemma by multiplying this last inequality by $e^{-\lambda t}$ and by integrating between 0 and $+\infty$.

Now, we know that

$$\varphi \in \mathbf{C}_b(X) \Rightarrow T_\lambda \varphi \in \mathbf{C}_b^{1+\varepsilon_0}(X)$$

$$\varphi \in \mathbf{C}_b^1(X) \Rightarrow T_\lambda \varphi \in \mathbf{C}_b^2(X).$$

Again, by interpolating, we have, for all $\alpha \in (0, 1)$,

$$\varphi \in \mathbf{C}_b^\alpha(X) \Rightarrow T_\lambda \varphi \in \mathbf{C}_b^{(1+\varepsilon_0)(1-\alpha)+2\alpha}(X)$$

and (ii) is proven. Finally, let $\varphi \in \mathbf{C}_b^1(X)$; we have

$$\|(P_t \varphi)_{xx}\| \leq C_0 \|\Gamma(t)\|^2 \|\varphi\|$$

$$\|(P_t \varphi)_{xx}\| \leq C_1 \|\Gamma(t)\| \|\varphi\|_1.$$

thus, for all $\theta \in (0, 1)$, we have

$$\|(P_t \varphi)_{xx}\| \leq C_\theta \|\Gamma(t)\|^{2-\theta} \|\varphi\|_\theta, \quad \forall \varphi \in \mathbf{C}_b^\gamma(X), \quad \forall t > 0.$$

By choosing, $\theta = 1 - \varepsilon_0$, we get, for all $\varphi \in \mathbf{C}_b^{1-\varepsilon_0}(X)$,

$$\|(P_t \varphi)_{xx}\| \leq C_{1-\varepsilon_0} \|\Gamma(t)\|^{1+\varepsilon_0} \|\varphi\|_{1-\varepsilon_0}$$

and we conclude as previously in order to get (iii), which completes the proof of our lemma. ■

Proof of Theorem 3.3(ii). Now, let ψ be in $\mathbf{C}_b^1(X)$ and $\lambda \geq \lambda_0$. Then, ψ belongs to $\mathbf{C}_b^\alpha(X)$ for all $\alpha \in [0, 1]$. Moreover if $u \in \mathbf{C}_b^{1+\alpha}(X)$ for some $\alpha \in [0, 1)$, since $\psi + \langle F, u_x \rangle - H(u_x) \in \mathbf{C}_b^\alpha(X)$ and $u = T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)]$, we have by Lemma 3.6(ii) $u \in \mathbf{C}_b^{1+\varepsilon_0+\alpha-\alpha\varepsilon_0}(X)$. By a bootstrap argument, we prove that u belongs to $\mathbf{C}_b^{1+\alpha_n}(X)$ for all $n \in \mathbb{N}$, where the sequence $(\alpha_n)_{n \in \mathbb{N}}$ is defined by

$$\alpha_0 = 0$$

$$\alpha_{n+1} = \varepsilon_0 + \alpha_n - \varepsilon_0 \alpha_n.$$

We check easily that $(\alpha_n)_{n \in \mathbb{N}}$ converges to 1 as n goes to infinity. Therefore, we can prove that $u \in \mathbf{C}_b^{2-\varepsilon_0}(X)$, which yields, by the third point of the

previous lemma, $u \in \mathbf{C}_b^2(X)$ and the second item of the theorem is proven. ■

Proof of Theorem 3.3(iv). We now prove (3.4). Let $\lambda, \mu \geq \lambda_0$, $\psi \in \mathbf{C}_b(X)$. Recall that T_λ satisfies

$$T_\lambda = T_\mu o[1 + (\mu - \lambda) T_\lambda].$$

If we set $u = F_\lambda(\psi)$, the identity (3.4) is equivalent to

$$T_\mu^{\psi + (\mu - \lambda)u}(u) = u.$$

But

$$\begin{aligned} T_\mu^{\psi + (\mu - \lambda)u}(u) &= T_\mu[\psi + (\mu - \lambda)u + \langle F, u_x \rangle - H(u_x)] \\ &= T_\mu[\psi + \langle F, u_x \rangle - H(u_x) \\ &\quad + (\mu - \lambda) T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)]] \\ &= T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)] = u \end{aligned}$$

and the proof of (3.4) is complete. ■

We shall first prove part (iii) of Theorem 3.3 for $\varphi, \psi \in \mathbf{C}_b^1(X)$. Then it will be easy to extend (3.3) to functions in $\mathbf{C}_b(X)$ by using the following lemma:

LEMMA 3.7. *Assume that (3.3) holds for all $\psi, \varphi \in \mathbf{C}_b^1(X)$. Then it also holds for $\psi, \varphi \in \mathbf{C}_b(X)$.*

Proof. Let $\lambda \geq \lambda_0$ and $\varphi, \psi \in \mathbf{C}_b(X)$. Then there exist two sequences $(\varphi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ of $\mathbf{C}_b^1(X)$ which converge respectively towards φ and ψ in $\mathbf{C}_b(X)$ (see [22] and [25]). Let

$$u = F_\lambda(\psi), \quad u_n = F_\lambda(\psi_n), \quad v = F_\lambda(\varphi), \quad v_n = F_\lambda(\varphi_n),$$

for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we have

$$\|u - v\| \leq \|u - u_n\|_1 + \|u_n - v_n\| + \|v_n - v\|_1.$$

But we have, for instance,

$$\|u - u_n\|_1 \leq \|T_\lambda^\psi u - T_\lambda^\psi u_n\|_1 + \|T_\lambda^\psi u_n - T_\lambda^{\psi_n} u_n\|_1$$

and, by using Propositions 3.4 and 3.5, it yields

$$\|u - u_n\|_1 \leq \gamma \|u - u_n\|_1 + \gamma' \|\psi - \psi_n\|$$

where

$$\gamma' = \frac{1}{\lambda} + \alpha(\lambda), \quad \gamma = \gamma'(\|F\| + K) < 1.$$

Thus,

$$\|u - u_n\|_1 \leq \frac{\gamma'}{1 - \gamma} \|\psi - \psi_n\|$$

and

$$\|u - v\| \leq \frac{\gamma'}{1 - \gamma} (\|\psi_n - \psi\| + \|\varphi_n - \varphi\|) + \frac{1}{\lambda} \|\psi_n - \varphi_n\|$$

since φ_n and ψ_n belong to $\mathbf{C}_b^1(X)$. Thus, when n goes to infinity, we have

$$\|u - v\| \leq \frac{1}{\lambda} \|\psi - \varphi\|. \quad \blacksquare$$

Proof of Theorem 3.3(iii) in $\mathbf{C}_b^1(X)$. We now turn back to the case when φ and ψ belong to $\mathbf{C}_b^1(X)$. Let $\lambda \geq \lambda_0$. We set $u = F_\lambda(\psi)$ and $v = F_\lambda(\varphi)$. It is clear that if we try to compute directly $\|u - v\|$, we don't get the desired estimation since T_λ^ψ and T_λ^φ are nonlinear. Thus, we have to approximate the nonlinear term. This is the aim of the next lemma that we shall prove later on.

LEMMA 3.8. *There exists a sequence of operators $(N_\varepsilon)_{\varepsilon \geq 0}$ which satisfies:*

(i) *for all $w_1, w_2 \in \mathbf{C}_b(X)$,*

$$\|N_\varepsilon w_1 - N_\varepsilon w_2\| \leq \|w_1 - w_2\|, \quad \forall \varepsilon \geq 0. \quad (3.5)$$

(ii) *for all $w \in \mathbf{C}_b^2(X)$,*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{N_\varepsilon w - w}{\varepsilon} - \langle F, w_x \rangle + H(w_x) \right\| = 0. \quad (3.6)$$

We set, for all $\varepsilon > 0$ and all $w \in \mathbf{C}_b^2(X)$,

$$\psi_\varepsilon(w) = \frac{N_\varepsilon w - w}{\varepsilon} - \langle F, w_x \rangle + H(w_x).$$

We first notice that, for $\varepsilon > 0$, we have, by setting $\mu = \lambda + 1/\varepsilon$ in (3.4),

$$u = F_{\lambda + (1/\varepsilon)} \left(\psi + \frac{1}{\varepsilon} u \right) = T_{\lambda + (1/\varepsilon)} \left[\psi + \frac{1}{\varepsilon} u + \langle F, u_x \rangle - H(u_x) \right]$$

and an equivalent identity for v . It yields

$$u - v = T_{\lambda + (1/\varepsilon)} \left[\psi - \varphi + \frac{1}{\varepsilon} (N_\varepsilon u - N_\varepsilon v) - \psi_\varepsilon(u) + \psi_\varepsilon(v) \right]$$

and thus, by using (3.5),

$$\|u - v\| \leq \frac{1}{\lambda + (1/\varepsilon)} \left(\|\psi - \varphi\| + \frac{1}{\varepsilon} \|u - v\| + \|\psi_\varepsilon(u)\| + \|\psi_\varepsilon(v)\| \right).$$

This implies

$$\lambda \|u - v\| \leq \|\psi - \varphi\| + \|\psi_\varepsilon(u)\| + \|\psi_\varepsilon(v)\|.$$

By letting ε go to 0 and by using (3.6), we get (3.3). ■

Now, we turn back to the proof of Lemma 3.8. Roughly speaking, $(N_\varepsilon)_{\varepsilon \geq 0}$ has to be the semigroup associated to the equation

$$W_t = \langle F, W_x \rangle - H(W_x)$$

$$W(0) = w$$

which is a time-dependent first-order Hamilton–Jacobi equation whose solution should be the value function of the following optimal control problem: the dynamic of the system is described by the equation

$$\dot{y}_x(s) = F(y_x(s)) - \alpha(s), \quad s > 0$$

$$y_x(0) = x.$$

The controls α are taken in the set

$$\mathcal{A} = \{ \alpha \in W^{1, \infty}((0, +\infty); X) \text{ s.t. } |\alpha(s)| \leq M \forall s \in [0, +\infty) \}$$

$$\text{and } |\dot{\alpha}(s)| \leq 1 \text{ a.e. } s \in (0, +\infty) \}$$

where $M > 0$ is a constant that we will precise later. The cost function is given by

$$J(x, t, \alpha) = \int_0^t g(\alpha(s)) ds + w(y_x(t)),$$

where the function g is defined on X by

$$g(\alpha) = \sup_{|p| \leq M} \{ \langle \alpha, p \rangle - H(p) \}$$

and the value function is

$$W(x, t) = \inf_{\alpha \in \mathcal{A}} J(x, t, \alpha).$$

We are not going to prove this result because we only use it in an heuristic way but one can see [11] for precise results on first-order Hamilton–Jacobi equations and deterministic control.

Now, for $w \in C_b(X)$, we set $N_\varepsilon w = W(., \varepsilon)$ and we prove that it satisfies the properties listed in Lemma 3.8.

Proof of Lemma 3.8. Let $w_1, w_2 \in C_b(X)$. For all $\varepsilon > 0$ and for all $x \in X$, we have

$$\begin{aligned} |N_\varepsilon w_1(x) - N_\varepsilon w_2(x)| &\leq \sup_{\alpha \in \mathcal{A}} \{w_1(y_x(\varepsilon)) - w_2(y_x(\varepsilon))\} \\ &\leq \|w_1 - w_2\|. \end{aligned}$$

In order to prove (3.6), we first admit the following lemma.

LEMMA 3.9. *If $M \geq K$ then, for all $p \in X$ such that $|p| \leq M$, we have*

$$H(p) = \sup_{|\alpha| \leq M} \{ \langle p, \alpha \rangle - g(\alpha) \}.$$

Now, let $u \in C_b^2(X)$ and chose a constant $M \geq \max\{\|u_x\|, K\}$. Let $\varepsilon > 0$ and $x \in X$. It is clear that the value function is limited by the infimum over all constant controls of the cost function. And by using Lemma 3.9, we have

$$\begin{aligned} \psi_\varepsilon(u)(x) &\leq \inf_{|\alpha| \leq M} \left\{ g(\alpha) + \frac{u(y_x(\varepsilon)) - u(x)}{\varepsilon} \right\} \\ &\quad - \inf_{|\alpha| \leq M} \{ \langle F(x) - \alpha, u_x(x) \rangle + g(\alpha) \} \\ &\leq \sup_{|\alpha| \leq M} \left\{ \frac{u(y_x(\varepsilon)) - u(x)}{\varepsilon} - \langle F(x) - \alpha, u_x(x) \rangle \right\}. \end{aligned}$$

We write

$$u(y_x(\varepsilon)) - u(x) = \int_0^\varepsilon \langle F(y_x(\tau)) - \alpha, u_x(y_x(\tau)) \rangle d\tau$$

and we get

$$\begin{aligned} \psi_\varepsilon(u)(x) &\leq \sup_{|\alpha| \leq M} \left\{ \frac{1}{\varepsilon} \int_0^\varepsilon \langle F(y_x(\tau)) - \alpha, u_x(y_x(\tau)) \rangle \right. \\ &\quad \left. - \langle F(x) - \alpha, u_x(x) \rangle \right\} d\tau \\ &\leq \sup_{|\alpha| \leq M} \left\{ \frac{1}{\varepsilon} \int_0^\varepsilon [\langle F(y_x(\tau)) - F(x), u_x(y_x(\tau)) \rangle \right. \\ &\quad \left. + \langle F(x) - \alpha, u_x(y_x(\tau)) - u_x(x) \rangle] d\tau \right\}. \end{aligned}$$

But, since, for all $\tau \in (0, \varepsilon)$,

$$|y_x(\tau) - x| = |y_x(\tau) - y_x(0)| \leq (\|F\| + M)\tau,$$

we have

$$\psi_\varepsilon(u)(x) \leq \frac{1}{\varepsilon} \int_0^\varepsilon \|u_x\| L(\|F\| + M) \tau d\tau + \frac{\|F\| + M}{\varepsilon} \int_0^\varepsilon \|u_{xx}\| (\|F\| + M) \tau d\tau$$

and finally,

$$\psi_\varepsilon(u)(x) \leq (\|F\| + M)(\|u_x\| L + \|u_{xx}\|(\|F\| + M)) \frac{\varepsilon}{2}.$$

We now deal with the converse inequality. For all $\delta > 0$, there exists $\alpha_\delta \in \mathcal{A}$ (which in fact depends also on ε and x) such that

$$N_\varepsilon u(x) \leq \int_0^\varepsilon g(\alpha_\delta(s)) ds + u(y_x^\delta(\varepsilon)) \leq N_\varepsilon u(x) + \delta,$$

where y_x^δ is the state which corresponds to α_δ .

Therefore

$$\begin{aligned} \psi_\varepsilon(u)(x) &\geq \frac{1}{\varepsilon} \int_0^\varepsilon g(\alpha_\delta(s)) ds + \frac{u(y_x^\delta(\varepsilon)) - u(x)}{\varepsilon} \\ &\quad - \langle F(x), u_x(x) \rangle + H(u_x(x)) - \frac{\delta}{\varepsilon}. \end{aligned}$$

But $H(u_x(x)) \geq \langle \alpha_\delta(0), u_x(x) \rangle - g(\alpha_\delta(0))$, thus

$$\begin{aligned} \psi_\varepsilon(u)(x) &\geq \frac{1}{\varepsilon} \int_0^\varepsilon g(\alpha_\delta(s)) ds + \frac{u(y_x^\delta(\varepsilon)) - u(x)}{\varepsilon} \\ &\quad + \langle \alpha_\delta(0) - F(x), u_x(x) \rangle - g(\alpha_\delta(0)) - \frac{\delta}{\varepsilon} \\ &\geq \frac{1}{\varepsilon} \int_0^\varepsilon (g(\alpha_\delta(s)) - g(\alpha_\delta(0))) ds + \frac{u(y_x^\delta(\varepsilon)) - u(x)}{\varepsilon} \\ &\quad + \langle \alpha_\delta(0) - F(x), u_x(x) \rangle - \frac{\delta}{\varepsilon}. \end{aligned}$$

Since

$$|g(\alpha) - g(\beta)| \leq \sup_{|p| \leq M} \langle \alpha - \beta, p \rangle \leq M |\alpha - \beta| \quad \forall \alpha, \beta \in X,$$

we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^\varepsilon (g(\alpha_\delta(s)) - g(\alpha_\delta(0))) ds &\geq -\frac{M}{\varepsilon} \int_0^\varepsilon |\alpha_\delta(s) - \alpha_\delta(0)| ds \\ &\geq -\frac{M}{\varepsilon} \int_0^\varepsilon s ds = -\frac{M\varepsilon}{2}. \end{aligned}$$

We write

$$u(y_x^\delta(\varepsilon)) - u(x) = \int_0^\varepsilon \langle F(y_x^\delta(\tau)) - \alpha_\delta(\tau), u_x(y_x^\delta(\tau)) \rangle d\tau;$$

then

$$\begin{aligned} &\frac{u(y_x^\delta(\varepsilon)) - u(x)}{\varepsilon} + \langle \alpha_\delta(0) - F(x), u_x(x) \rangle \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon \langle F(y_x^\delta(\tau)) - F(x) + \alpha_\delta(0) - \alpha_\delta(\tau), u_x(y_x^\delta(\tau)) \rangle d\tau \\ &\quad + \frac{1}{\varepsilon} \int_0^\varepsilon \langle F(x) - \alpha_\delta(0), u_x(y_x^\delta(\tau)) - u_x(x) \rangle d\tau \\ &\geq -\frac{1}{\varepsilon} \int_0^\varepsilon \|u_x\| (L(\|F\| + M) + 1) \tau d\tau \\ &\quad - \frac{1}{\varepsilon} \int_0^\varepsilon (\|F\| + M) \|u_{xx}\| (\|F\| + M) \tau d\tau \\ &\geq -(\|u_x\| (L(\|F\| + M) + 1) + \|u_{xx}\| (\|F\| + M)^2) \frac{\varepsilon}{2} \end{aligned}$$

Finally,

$$|\psi_\varepsilon(u)(x)| \leq (M + \|u_x\|(L(\|F\| + M) + 1) + \|u_{xx}\|(\|F\| + M)^2) \frac{\varepsilon}{2} + \frac{\delta}{\varepsilon}, \quad \forall x \in X, \quad \forall \delta > 0$$

therefore

$$\|\psi_\varepsilon(u)\| \leq (M + \|u_x\|(L(\|F\| + M) + 1) + \|u_{xx}\|(\|F\| + M)^2) \frac{\varepsilon}{2}$$

and goes to 0 as ε goes to 0. ■

Proof of Lemma 3.9. Let G be defined on X by

$$G(p) = \sup_{|\alpha| \leq M} \{ \langle p, \alpha \rangle - g(\alpha) \}$$

and let $p_0 \in X$ be such that $|p_0| \leq M$. Then

$$G(p_0) = \sup_{|\alpha| \leq M} \inf_{|p| \leq M} \{ \langle p_0 - p, \alpha \rangle + H(p) \} \leq H(p_0)$$

by choosing $p = p_0$. Moreover,

$$G(p_0) = H(p_0) + \sup_{|\alpha| \leq M} \inf_{|p| \leq M} \{ \langle p_0 - p, \alpha \rangle + H(p) - H(p_0) \}.$$

Since H is convex, we have $H(p) - H(p_0) \geq \langle q_0, p - p_0 \rangle$ for all q_0 in the subdifferential of H at p_0 . Thus

$$G(p_0) \geq H(p_0) + \sup_{|\alpha| \leq M} \inf_{|p| \leq M} \{ \langle p_0 - p, \alpha \rangle + \langle q_0, p - p_0 \rangle \}$$

and

$$G(p_0) \geq H(p_0)$$

by choosing $\alpha = q_0$, which is possible since $|q_0| \leq K \leq M$. ■

3.2. Characterization of the Nonlinear Operator

In order to deduce, from the preceding section, the existence (and uniqueness) of a nonlinear operator defined on a domain of $C_b(X)$, which generates the solution of (3.1), we shall use the results contained in [12].

We proved that there exists a $\lambda_0 > 0$ such that the application \mathcal{F} defined by $\mathcal{F}(\lambda) = F_\lambda$ for all $\lambda \geq \lambda_0$ takes its values in the set of the Lipschitz-continuous functions defined on $\mathbf{C}_b(X)$, since

$$\|F_\lambda \psi - F_\lambda \varphi\| \leq \frac{1}{\lambda} \|\psi - \varphi\|, \quad \forall \lambda \geq \lambda_0, \quad \forall \psi, \varphi \in \mathbf{C}_b(X). \quad (3.7)$$

Moreover, this application satisfies, for all $\lambda, \mu \geq \lambda_0$

$$F_\lambda = F_\mu o(1 + (\mu - \lambda) F_\lambda). \quad (3.8)$$

Thus, there exists a unique operator \mathcal{B} defined on $D(\mathcal{B}) \subset \mathbf{C}_b^1(X)$ and which takes its values in $\mathbf{C}_b(X)$ such that $F_\lambda = R(\lambda, \mathcal{B})$ for all $\lambda \geq \lambda_0$. In particular, the resolvent set of \mathcal{B} , say $\rho(\mathcal{B})$, contains $[\lambda_0, +\infty)$.

These properties imply that \mathcal{B} is an accretive operator and that $\rho(\mathcal{B})$ contains $(0, +\infty)$. Moreover the properties of \mathcal{F} , namely (3.7) and (3.8), hold for $R(\cdot, \mathcal{B})$, for any $\lambda, \mu > 0$.

Since the image of $\mathbf{C}_b(X)$ by $R(\lambda, \mathcal{B})$ is constant, we define $D(\mathcal{B}) = R(\lambda, \mathcal{B}) \mathbf{C}_b(X)$ for any $\lambda > 0$ and \mathcal{B} is defined by $\mathcal{B}u = \lambda u - \psi$ for some $\psi \in \mathbf{C}_b(X)$.

Again, we can show that if $\psi \in \mathbf{C}_b^1(X)$ then $R(\lambda, \mathcal{B}) \psi \in \mathbf{C}_b^2(X)$, even for small $\lambda > 0$. Indeed we have the following proposition.

PROPOSITION 3.10. *For all $\lambda > 0$, for all $\psi \in \mathbf{C}_b(X)$, $R(\lambda, \mathcal{B})\psi$ is a fixed point of T_λ^ψ . Conversely if $u \in \mathbf{C}_b^1(X)$ is a fixed point of T_λ^ψ , then $u = R(\lambda, \mathcal{B})\psi$.*

Proof. Recall that, for all $\lambda > 0$, $F_\lambda(\psi)$ is the set of all fixed points of T_λ^ψ . Let $u = R(\lambda, \mathcal{B})\psi$. Then $u = R(\lambda + \lambda_0, \mathcal{B})(\psi + \lambda_0 u) = F_{\lambda + \lambda_0}(\psi + \lambda_0 u)$ which is the fixed point of $T_{\lambda + \lambda_0}^{\psi + \lambda_0 u}$. Thus

$$\begin{aligned} u &= T_{\lambda + \lambda_0}[\psi + \lambda_0 u + \langle F, u_x \rangle - H(u_x)] \\ &= T_\lambda[\psi + \lambda_0 u + \langle F, u_x \rangle - H(u_x) + (\lambda - \lambda - \lambda_0)u] \\ &= T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)] = T_\lambda^\psi(u), \end{aligned}$$

and $u \in F_\lambda(\psi)$.

Conversely, if u is a fixed point of T_λ^ψ , then $u \in F_\lambda(\psi)$. We can prove, as we did for Theorem 3.3(iv), that this implies $u \in F_{\lambda + \lambda_0}(\psi + \lambda_0 u)$. But

$$F_{\lambda + \lambda_0}(\psi + \lambda_0 u) = R(\lambda + \lambda_0, \mathcal{B})(\psi + \lambda_0 u) = R(\lambda, \mathcal{B})\psi$$

and the proof is complete. \blacksquare

We summarize those results in the

THEOREM 3.11. $\forall \lambda > 0, \forall \psi \in \mathbf{C}_b^1(X)$, there exists a unique $u \in \mathbf{C}_b^1(X)$ mild solution of (3.1). Moreover, if $\psi \in \mathbf{C}_b^1(X)$, then $u \in \mathbf{C}_b^2(X)$.

Proof. The existence and uniqueness of the mild solution of (3.1) follows directly from the results stated above. The rest is straightforward since $R(\lambda, \mathcal{B})\psi$ is a fixed point of T_λ^ψ and since we never used the fact that λ was greater than λ_0 in the proof of part (ii) of Theorem 3.3 except to deduce from it that some fixed point exists. ■

We continue to investigate the properties of \mathcal{B} .

PROPOSITION 3.12. *The operator \mathcal{B} is monovalued.*

Proof. Indeed, let $u \in D(\mathcal{B})$ and $\varphi, \psi \in \mathcal{B}u$. Then, for all $\lambda > 0$, $\lambda u - \varphi$ and $\lambda u - \psi$ belong to $\lambda u - \mathcal{B}u$, i.e. $u = R(\lambda, \mathcal{B})(\lambda u - \psi) = R(\lambda, \mathcal{B})(\lambda u - \varphi)$. By using Proposition 3.10, it yields

$$T_\lambda[\lambda u - \psi + \langle F, u_x \rangle - H(u_x)] = T_\lambda[\lambda u - \varphi + \langle F, u_x \rangle - H(u_x)]$$

and thus we have

$$T_\lambda \varphi(x) = T_\lambda \psi(x), \quad \forall \lambda > 0 \quad \text{and} \quad \forall x \in X.$$

By using a well-known property of the Laplace transform, we deduce that for all $x \in X$, $P_t \varphi(x) = P_t \psi(x)$ for all $t > 0$. Now, $(P_t)_{t>0}$ is not a strongly continuous semigroup but however $P_t \varphi(x)$ (for instance) converges, for each fixed x to $\varphi(x)$ as t goes to 0 and thus $\varphi(x) = \psi(x)$ for all $x \in X$, which concludes the proof. ■

Finally, to conclude this section, we characterize the operator \mathcal{B} thanks to the linear operator \mathcal{L} which has been studied previously (see e.g. [16] Ch. 9, [6] and [7]) and which consists of the linear part of the Hamilton–Jacobi equation.

THEOREM 3.13. *The following assertions hold:*

- (i) $D(\mathcal{B}) = D(\mathcal{L})$
- (ii) $\mathcal{B} = \mathcal{L} + \langle F, D_x \cdot \rangle - H(D_x \cdot)$.

Proof. We can define $D(\mathcal{L})$ as the set of all functions $T_\lambda \psi$ for $\psi \in \mathbf{C}_b^1(X)$ and for an arbitrary $\lambda > 0$, so that $D(\mathcal{L})$ is a subset of $\mathbf{C}_b^1(X)$. Now, let $u \in D(\mathcal{B})$ and $\psi = \lambda u - \mathcal{B}u$. Then $u \in \mathbf{C}_b^1(X)$, $u = T_\lambda[\psi + \langle F, u_x \rangle - H(u_x)]$ and thus $u \in D(\mathcal{L})$.

Conversely, let $u \in D(\mathcal{L})$ and $u = T_\lambda \psi$ for some $\psi \in \mathbf{C}_b(X)$. Then

$$\begin{aligned} u &= T_\lambda[\psi - \langle F, u_x \rangle + H(u_x) + \langle F, u_x \rangle - H(u_x)] \\ &= T_\lambda^{\psi - \langle F, u_x \rangle + H(u_x)}(u) \end{aligned}$$

and thus $u \in D(\mathcal{B})$.

Now, if $u \in D(\mathcal{B})$, let $\psi = \mathcal{B}u$. We have, for a positive $\lambda > 0$, $\lambda u - \psi = \lambda u - \mathcal{B}u$, i.e.

$$u = T_\lambda^{\lambda u - \psi}(u) = T_\lambda[\lambda u - \psi + \langle F, u_x \rangle - H(u_x)]$$

thus

$$\lambda u - \mathcal{L}u = \lambda u - \psi + \langle F, u_x \rangle - H(u_x)$$

and

$$\psi = \mathcal{L}u + \langle F, u_x \rangle - H(u_x)$$

which concludes the proof. \blacksquare

4. STRONG SOLUTIONS

Now we apply some results contained in [7] about Cauchy problems associated to weakly continuous semigroups to show that the mild solutions of (3.1) can be approximated by classical solutions. We first recall the definition of the \mathcal{K} -convergence introduced in [7].

DEFINITION 4.1. (i) A sequence $(\varphi_n) \subset \mathbf{C}_b(X, Y)$ is said to be \mathcal{K} -convergent to $\varphi \in \mathbf{C}_b(X, Y)$ if

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|\varphi_n\|_0 &< +\infty \\ \lim_{n \rightarrow +\infty} \sup_{x \in K} |\varphi_n(x) - \varphi(x)|_Y &= 0. \end{aligned}$$

for every compact set $K \subset X$. In this case we shall write

$$\varphi = \mathcal{K}\text{-}\lim_{n \rightarrow +\infty} \varphi_n.$$

(ii) A linear operator $A: D(A) \subset \mathbf{C}_b(X) \rightarrow \mathbf{C}_b(X)$ is said to be \mathcal{K} -closed if, given a sequence $(\varphi_n)_n \subset D(A)$ such that

$$\mathcal{K}\text{-}\lim_{n \rightarrow +\infty} \varphi_n = \varphi \quad \text{and} \quad \mathcal{K}\text{-}\lim_{n \rightarrow +\infty} A\varphi_n = \psi,$$

we have

$$\varphi \in D(A) \quad \text{and} \quad A\varphi = \psi.$$

(iii) Let $A: D(A) \subset \mathbf{C}_b(X) \rightarrow \mathbf{C}_b(X)$ and $C: D(C) \subset \mathbf{C}_b(X) \rightarrow \mathbf{C}_b(X)$ be two linear operators; assume that $A \subset C$ and that C is \mathcal{K} -closed. We say that C is the \mathcal{K} -closure of A , and we write $C = \bar{A}^{\mathcal{K}}$, if for every $\varphi \in D(C)$ there exists a sequence $(\varphi_n)_n \subset D(A)$ such that

$$\begin{aligned} \mathcal{K}\text{-}\lim_{n \rightarrow +\infty} \varphi_n &= \varphi \\ \mathcal{K}\text{-}\lim_{n \rightarrow +\infty} A\varphi_n &= C\varphi. \end{aligned} \quad (4.1)$$

Now we recall that, the family of operators $\{P_t\}_{t>0}$ is a weakly continuous semigroup on $C_b(X)$ (see [6] and [7]). Let \mathcal{L} be the infinitesimal generator of $\{P_t; t \geq 0\}$. Following [7] we define the operator \mathcal{L}_0 as follows:

$$\begin{aligned} D(\mathcal{L}_0) &= \{\varphi \in C_b^2(X) : \text{Tr } \varphi_{xx} \in C_b(X), A^*\varphi_x \in C_b(X, X); \\ &\quad x \rightarrow \langle x, A^*\varphi_x(x) \rangle \in C_b(X)\} \\ \mathcal{L}_0\varphi &= \frac{1}{2} \text{Tr}[\mathcal{Q}\varphi_{xx}] + \langle x, A^*\varphi_x \rangle. \end{aligned}$$

It is possible to see that $D(\mathcal{L}_0)$, endowed with the norm

$$\|\eta\|_* \stackrel{\text{def}}{=} \|\eta\| + \|A^*\eta_x\| + \|\langle \cdot, A^*\eta_x \rangle\| + \sup_{x \in X} \|\text{Tr } \eta_{xx}(x)\|$$

is a Banach space. Moreover the following result holds.

PROPOSITION 4.2. *Assume that Hypothesis 2.1, 2.2, 2.5 hold true. Then*

$$\overline{\mathcal{L}_0}^{\mathcal{K}} = \mathcal{L}. \quad (4.2)$$

Moreover for every $u \in D(\mathcal{L})$ there exists a sequence $(u_n)_n \subset D(\mathcal{L}_0)$ such that (4.1) is satisfied and

$$\mathcal{K}\text{-}\lim_{n \rightarrow +\infty} u_{nx} = u_x. \quad (4.3)$$

Proof. The proof of (4.2) is contained in [7] §5. We prove only (4.3). Let $u \in D(\mathcal{L})$ and let $\psi = (\lambda - \mathcal{L})u \in C_b(X)$. Then there exists a sequence $(\psi_n)_{n \in \mathbb{N}} \subset D(\mathcal{L}_0)$ such that $\psi_n \xrightarrow{\mathcal{K}} \psi$ as $n \rightarrow +\infty$. We set

$$u_n(x) = R(\lambda, \mathcal{L}) \psi_n(x) = \int_0^{+\infty} e^{-\lambda t} P_t \psi_n(x) dt.$$

The sequence $(u_n)_n$ lies in $D(\mathcal{L}_0)$ since

$$R(\lambda, \mathcal{L})D(\mathcal{L}_0) \subset D(\mathcal{L}_0)$$

and satisfies (4.1) (see [7]). Moreover, by differentiating we can write

$$u_{nx}(x) = \int_0^{+\infty} e^{-\lambda t} D_x [P_t \psi_n](x) dt$$

and, by using the explicit formula for the derivative $D_x P_t \psi_n$ (see e.g. [16] p. 264) we easily obtain that for every $t > 0$

$$D_x P_t \psi_n \xrightarrow{\mathcal{K}} D_x P_t \psi \quad \text{as } n \rightarrow \infty$$

and

$$\|D_x P_t \psi_n\| \leq C \|F(t)\|$$

for a positive constant C independent of n . Then the claim follows easily by applying the Dominated Convergence Theorem and the integrability of the map $t \rightarrow \|F(t)\|$. ■

By reasoning as in [18] we give the following definitions for solutions of the Hamilton–Jacobi equation (3.1).

DEFINITION 4.3. A function $u: X \rightarrow \mathbb{R}$ is a strict solution of the equation (3.1) if $u \in D(\mathcal{L}_0)$ and satisfies (3.1).

DEFINITION 4.4. A function $u \in C_b^1(X)$ is a \mathcal{K} -strong solution of equation (3.1) if there exist two sequences $\{u_n\} \subset D(\mathcal{L}_0)$ and $\{\psi_n\} \subset C_b(X)$ such that for every $n \in \mathbb{N}$, u_n is a strict solution of the problem:

$$\lambda u_n = \mathcal{L}_0 u_n + \langle F, u_{nx} \rangle - H(u_{nx}) + \psi_n \quad (4.4)$$

and moreover, for $n \rightarrow +\infty$

$$\begin{aligned} \psi_n &\xrightarrow{\mathcal{K}} \psi \\ u_n &\xrightarrow{\mathcal{K}} u \\ u_{nx} &\xrightarrow{\mathcal{K}} u_x. \end{aligned} \quad (4.5)$$

Now we apply Proposition 4.2 to equation (3.1) to obtain

THEOREM 4.5. *Assume that Hypotheses 2.1, 2.2, 2.5 and 3.1 hold true and let $u \in C_b^1(X)$. Then*

- (i) *If u is a strict solution of (3.1) then it is also a mild solution.*
- (ii) *If u is a mild solution of (3.1) and $u \in D(\mathcal{L}_0)$ then u is also a strict solution.*
- (iii) *u is a mild solution of (3.1) if and only if it is a \mathcal{K} -strong solution.*

Proof. Statement (i) follows immediately from the definitions, while (ii) is a consequence of (4.2). We prove (iii) starting by the “only if” part.

Let $\psi \in C_b(X)$. Let $u = R(\lambda, \mathcal{B})\psi$ be the mild solution of (3.1). Then $u \in D(\mathcal{L})$ and by Proposition 4.2, there exists a sequence $(u_n)_n \subset D(\mathcal{L}_0)$ satisfying (4.1) and (4.3). Moreover, by setting

$$\psi_n = \lambda u_n - \mathcal{L}_0 u_n + H(u_{nx}) - \langle F, u_{nx} \rangle$$

we have that $\psi_n \in C_b(X)$ and (4.4), (4.5) are satisfied. This concludes the “only if” part.

To prove the “if” part, let u be a \mathcal{K} -strong solution and let $\{u_n\}_{n \in \mathbb{N}}$ be the approximating sequence as in Definition 4.4. Then for every $n \in \mathbb{N}$, u_n satisfies

$$\mathcal{L}_0 u_n = \lambda u_n - \langle F, u_{nx} \rangle + H(u_{nx}) - \psi_n,$$

and by (4.5) the right hand side \mathcal{K} -converges to $\lambda u - \langle F, u_x \rangle + H(u_x) - \psi$. By Proposition 4.2, it follows that $u \in D(\mathcal{L})$ and

$$\mathcal{L}u = \lambda u - \langle F, u_x \rangle + H(u_x) - \psi$$

which gives the claim. ■

5. APPLICATION TO A CONTROL PROBLEM

We consider a stochastic system governed by the state equation

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} [F(y(s)) - z(s)] ds + W_A(t), \quad t \geq 0 \quad (5.1)$$

where $x \in X$, A and F satisfy Hypotheses 2.1-(i) and 3.1-(ii) respectively. The controls z are taken in $M_{\mathcal{W}}^2(0, +\infty; X)$ and

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s).$$

Equation (5.1) can be viewed as the mild form of the stochastic differential equation

$$\begin{aligned} dy(s) &= [Ay(s) + F(y(s)) - z(s)] ds + \sqrt{Q} dW(s), \quad s \geq 0 \\ y(0) &= x, \quad x \in X \end{aligned} \quad (5.2)$$

where $W(\cdot)$ is a cylindrical Wiener process (see Hypothesis 2.1). The following Proposition is proved in [16], Ch. 7.1 and, in the case where A is diagonal, in [4].

PROPOSITION 5.1. *Assume that Hypothesis 2.1 and 2.2 and 3.1(ii) hold true. Then, for all $z \in M_{\mathcal{W}}^2(0, +\infty; X)$, Equation (5.1) has a unique solution*

$$(\cdot; x, z) \in M_{\mathcal{W}}^2(0, +\infty; X).$$

Moreover, if

$$\int_0^T s^{-\beta} \operatorname{Tr} e^{sA} Q e^{sA*} ds < +\infty$$

holds for some $\beta > 0$ and $T > 0$, then the solution $y(\cdot, x, z)$ is continuous \mathbf{P} -almost surely.

We shall now study the following optimal stochastic control problem. Given $R > 0$, minimize the functional cost

$$J(x; z) = \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} [\psi(y(s; x, z)) + h(z(s))] ds \right\}$$

over all controls $z \in M_{\mathcal{W}}^2(0, +\infty; X)$ satisfying $|z(s)| \leq R$ \mathbf{P} -almost surely for a.e. $s \in [0, +\infty)$. Here $\psi \in \mathbf{C}_b(X)$ and the function h satisfies the following assumption

HYPOTHESES 5.2. *$h: B(0, R) \subset X \rightarrow \mathbb{R}$ is convex and lower semicontinuous.*

The value function of this problem is defined as

$$V(x) = \inf \{ J(x; z) : z \in M_{\mathcal{W}}^2(0, +\infty; X), |z(s)| \leq R \} \quad (5.3)$$

and a control $z^* \in M_{\mathcal{W}}^2(0, +\infty; X)$ satisfying $|z^*(s)| \leq R$ and $V(x) = J(x, z^*)$ is said to be *optimal* with respect to the initial state x . As seen in the introduction, the corresponding Hamilton–Jacobi equation reads as follows

$$\lambda v = \frac{1}{2} \operatorname{Tr} [Q v_{xx}] + \langle Ax + F(x), v_x \rangle - H(v_x) + \psi(x), \quad x \in X \quad (5.4)$$

where the Hamiltonian H is given on X by

$$H(p) = \sup_{|z| \leq R} \{ \langle z, p \rangle - h(z) \}. \quad (5.5)$$

The aim of this section is to prove that, under the general assumption 5.2 on the cost h , the value function V is the mild solution of the Hamilton–Jacobi equation (5.4) and that, when H is smooth enough, there exists an optimal control.

We want to emphasize here that this result is very interesting in terms of optimal control since it states that the value function is smooth (at least $C_b^1(X)$) for general costs J with h satisfying 5.2 and ψ in $C_b(X)$.

THEOREM 5.3. *Assume that Hypotheses 2.1, 2.2, 2.5, 3.1-(ii) and 5.2 hold. Let $\psi \in C_b(X)$, H be as in (5.5) and let $u \in D(\mathcal{B})$ be the mild solution of (5.4). Then $u = V$ on X .*

The proof of this theorem involves two different types of arguments. In order to prove that $V \geq u$ for general h and ψ , we establish the fundamental equality (5.6) by using that u is a \mathcal{K} -strong solution as seen in the previous section.

Then, we can prove that the converse inequality holds in the case when h is smooth enough by exhibiting an optimal control under feedback form.

The final step consists in getting rid of the smoothness assumption on h ; it is done by approximating H with its Yosida's approximants.

We first start by the fundamental equality:

LEMMA 5.4. *Under the assumptions of the previous theorem, we have, for every $x \in X$ and $z \in M_w^2(0, +\infty; X)$ satisfying $|z(s)| \leq R$ \mathbf{P} -almost surely for a.e. s in $[0, +\infty)$,*

$$\begin{aligned} u(x) + \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} [H(u_x(y(s))) - \langle z(s), u_x(y(s)) \rangle + h(z(s))] ds \right\} \\ = \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} [\psi(y(s)) + h(z(s))] ds \right\} = J(x, z) \end{aligned} \quad (5.6)$$

where $y(s) \stackrel{\text{def}}{=} y(s; x, z)$ is the mild solution of (5.2).

Proof. We first observe that H is well defined and Lipschitz continuous with a Lipschitz constant lower than R . Indeed, by (5.5),

$$|H(p) - H(q)| \leq \sup_{|z| \leq R} \langle z, p - q \rangle \leq R |p - q|,$$

so that the results of the preceeding sections hold. Now let u be the solution of the Hamilton–Jacobi equation (5.4) and let u_n, ψ_n be as in Definition 4.4. We first prove that (5.6) holds for u_n , i.e. that

$$\begin{aligned} u_n(x) + \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} [H(u_{nx}(y(s))) - \langle z(s), u_{nx}(y(s)) \rangle + h(z(s))] ds \right\} \\ = \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} [\psi_n(y(s)) + h(z(s))] ds \right\}. \end{aligned} \quad (5.7)$$

Indeed fix $x \in X, z \in M_W^2(0, +\infty; X)$ and let $y = y(\cdot; x, z) \in M_W^2(0, +\infty; X)$ be the solution of the corresponding state equation. By applying Itô's formula to the process $e^{-\lambda t} u_n(y(t))$, we obtain

$$\begin{aligned} d[e^{-\lambda t} u_n(y(t))] = \left[-\lambda e^{-\lambda t} u_n(y(t)) + \frac{e^{-\lambda t}}{2} \text{Tr} [Q u_{nxx}(y(t))] \right] dt \\ + \langle dy(t), e^{-\lambda t} u_{nx}(y(t)) \rangle \end{aligned}$$

which gives, by (4.4),

$$\begin{aligned} d[e^{-\lambda t} u_n(y(t))] = e^{-\lambda t} [-\langle z(t), u_{nx}(y(t)) \rangle + H(u_{nx}(y(t))) - \psi_n(y(t))] dt \\ + e^{-\lambda t} \langle \sqrt{Q} dW(t), u_{nx}(y(t)) \rangle. \end{aligned}$$

Then (5.7) follows easily by adding $h(z(s))$ on both sides, by integrating on $[0, +\infty[$ and finally by taking the expectation.

Now, recall that (4.5) holds; so that by the Dominated Convergence Theorem, we can take the limit for $n \rightarrow +\infty$ in (5.7) in order to obtain the claim of the lemma. ■

COROLLARY 5.5. *Under the hypotheses of the previous lemma, we have:*

$$V \geq u \text{ on } X. \quad (5.8)$$

Proof. By the definition of H , for every $z \in M_W^2(0, +\infty; X)$, $x \in X$ and for a.e. $t \in [0, \infty[$, the following inequality holds \mathbf{P} -almost surely

$$H(u_x(y(t; x, z))) - \langle z(t), u_x(y(t; x, z)) \rangle + h(z(t)) \geq 0 \quad (5.9)$$

and the result follows from (5.6). ■

Remark. 5.6. The fundamental identity (5.6) and so (5.8) also hold when h is only measurable and bounded from below.

We can now prove that there exists an optimal control when H is smooth enough.

THEOREM 5.7. *Assume that Hypotheses 2.1, 2.2, 2.5, 3.1-(ii) and 5.2 hold true. We also assume that H is Gateaux differentiable with continuous directional derivatives. Finally, let ψ be in $\mathbf{C}_b(X)$. Then*

- (i) *the unique mild solution u of (5.4) coincides with the value function V given in (5.3);*
- (ii) *for any $x \in X$, there exists an optimal control z^* ;*
- (iii) *z^* is related to the corresponding optimal state y^* by the feedback formula*

$$z^*(t) = H_x(V_x(y^*(t))) \quad \forall t \geq 0;$$

- (iv) *if $H \in C^{1,1}(X)$ and $\psi \in \mathbf{C}_b^1(X)$ then the optimal control is unique.*

Proof. Let us first recall that, by the regularity of H , for every $p \in X$, the function $z \rightarrow \langle z, p \rangle - h(z)$ has its maximum on $B(0, R)$ at

$$z = H_x(p)$$

(see e.g. [17], § I.8). Then the equality in (5.9) holds when

$$z(t) = H_x(u_x(y(t; x, z))).$$

Now let us consider the closed loop equation

$$\begin{aligned} dy(s) &= [Ay(s) + F(y(s)) - H_x(u_x(y(s)))] ds + \sqrt{Q} dW(s), \quad s \geq 0 \\ y(0) &= x, \end{aligned} \quad (5.10) \quad x \in X.$$

which can be written in the mild form as

$$y(t) = e^{tA}x - \int_0^t e^{(t-s)A}[(F(y(s)) + H_x(u_x(y(s))))] ds + W_A(t), \quad t \geq 0.$$

Now, the regularity assumptions on H and the fact that $u \in \mathbf{C}_b^1(X)$ imply that the mapping

$$y \rightarrow \langle H_x(u_x(y)), h \rangle$$

is continuous and bounded on X for every $h \in X$. Since the semigroup e^{tA} is compact (see Remark 2.3-(i)), then, by a result of Chojnowska-Michalik and Goldys (see [8] Proposition 3), we obtain that equation (5.10) has a so-called martingale solution, which is mean square continuous and has a continuous modification.

At this point, by setting

$$z^*(s) = H_x(u_x(y^*(s))),$$

y^* is the mild solution of the state equation (5.1) when $z = z^*$ and the equality in (5.9) holds. By using (5.6) and Corollary 5.5, we have

$$V(x) \leq J(x, z^*) = u(x) \leq V(x)$$

which proves claims (i), (ii) and (iii).

To prove claim (iv), we only need to prove that when $H \in \mathbf{C}^{1,1}$ and $\psi \in \mathbf{C}_b^1(X)$, there exists a unique solution of the closed loop equation (5.10); in this case it is clear that the map

$$y \rightarrow H_x(u_x(y))$$

is Lipschitz continuous so that we can solve (5.10) by Proposition 5.1. The conclusion follows as in the previous case. ■

Remark 5.8. Examples for which H is Gateaux differentiable with continuous directional derivatives.

(i) The Hamiltonian has the above desired regularity if, for instance, $h: B(0, R) \subset X \rightarrow \mathbb{R}$ is strictly convex and lower semicontinuous.

(ii) A common example of the last statement is the following:

$$h(z) = |z|^\alpha,$$

for $\alpha > 1$. ■

Remark 5.9. Examples for which $H \in \mathbf{C}^{1,1}(X)$ holds.

(i) The Hamiltonian has the above desired regularity if, for instance, $h: B(0, R) \subset X \rightarrow \mathbb{R}$ is strictly convex and continuously Fréchet differentiable and if, moreover, $Dh: B(0, R) \rightarrow X$ is invertible with Lipschitz continuous inverse $(Dh)^{-1}$ (see [1], for instance).

(ii) A common example of the last statement is the following:

$$h(z) = \frac{1}{2} |z|^2.$$

In this case it is easy to check that

$$H(p) = \begin{cases} \frac{1}{2} |p|^2 & \text{if } |p| \leq R \\ R |p| - \frac{1}{2} R^2 & \text{if } |p| > R \end{cases}$$

and

$$H_x(p) = \begin{cases} p & \text{if } |p| \leq R \\ \frac{pR}{|p|} & \text{if } |p| > R. \end{cases}$$

In order to prove Theorem 5.3, we are going to use the Yosida's approximants of H which are given, for all $\varepsilon > 0$, by

$$H_\varepsilon(p) = \inf_{r \in X} \left\{ H(r) + \frac{1}{2\varepsilon} |p - r|^2 \right\}$$

and which satisfy the following lemma:

LEMMA 5.10. *For all $\varepsilon > 0$, we have:*

- (i) H_ε is convex and belongs to $\mathbf{C}^{1,1}(X)$
- (ii) $\|H_{\varepsilon X}\| \leq R$
- (iii) $0 \leq H(p) - H_\varepsilon(p) \leq \varepsilon R^2/2$ for all p in X .

Proof. The proofs of (i) and (ii) are contained, for instance, in [1]. It is clear that $H_\varepsilon(p) \leq H(p) + (1/2\varepsilon)|p - p|^2 = H(p)$ for all p in X . Now, let $p \in X$, we have

$$\begin{aligned} 0 \leq H(p) - H_\varepsilon(p) &= \sup_{r \in X} \left\{ H(p) - H(r) - \frac{1}{2\varepsilon} |p - r|^2 \right\} \\ &\leq \sup_{r \in X} \left\{ R |p - r| - \frac{1}{2\varepsilon} |p - r|^2 \right\}. \end{aligned}$$

The function $t \mapsto Rt - t^2/2\varepsilon$ reaches its maximum on \mathbb{R}^+ at εR and, as such, is bounded by $R(R\varepsilon) - (R\varepsilon)^2/2\varepsilon = \varepsilon R^2/2$ and the proof is complete. ■

Proof of Theorem 5.3. For all $\varepsilon > 0$, we define

$$h_\varepsilon(z) = \sup_{q \in X} \{ \langle q, z \rangle - H_\varepsilon(q) \} \quad \text{for all } z \in X$$

and

$$\tilde{H}_\varepsilon(p) = \sup_{|z| \leq R} \{ \langle p, z \rangle - h_\varepsilon(z) \} \quad \text{for all } p \in X.$$

Then $\tilde{H}_\varepsilon = H_\varepsilon$. Indeed, let $p \in X$; we have on one hand

$$\tilde{H}_\varepsilon(p) = \sup_{|z| \leq R} \inf_{q \in X} \{ \langle p - q, z \rangle + H_\varepsilon(q) \} \leq H_\varepsilon(p).$$

On the other hand, since H_ε is convex and differentiable, we have, for all $q \in X$,

$$H_\varepsilon(q) \geq H_\varepsilon(p) + \langle H_{\varepsilon X}(p), q - p \rangle.$$

Thus, for all $z \in X$,

$$\langle p - q, z \rangle + H_\varepsilon(q) \geq \langle p - q, z - H_{\varepsilon x}(p) \rangle + H_\varepsilon(p).$$

It yields

$$\tilde{H}_\varepsilon(p) \geq \sup_{|z| \leq R} \inf_{q \in X} \{ \langle p - q, z - H_{\varepsilon x}(p) \rangle \} + H_\varepsilon(p).$$

Since, $\|H_{\varepsilon x}\| \leq R$, the function on the right hand side is greater than its value at $z = H_{\varepsilon x}(p)$ and thus

$$\tilde{H}_\varepsilon(p) \geq H_\varepsilon(p).$$

We can now define a sequence of new optimal control problems in which the cost h has been replaced by h_ε and solve the associated Hamilton–Jacobi equations with the Hamiltonian $\tilde{H}_\varepsilon = H_\varepsilon$. Let V_ε and u_ε be respectively the corresponding value function and mild solution of the associated Hamilton–Jacobi equation. Since H_ε is smooth enough, then, by Theorem 5.7, we have $V_\varepsilon = u_\varepsilon$. We shall now see the relations between those functions and the corresponding ones for the initial problem. On one hand, as $H_\varepsilon \leq H$, we have, for all z such that $|z| \leq R$,

$$h_\varepsilon(z) \geq \sup_{p \in X} \{ \langle p, z \rangle - H(p) \} \geq \sup_{|p| \leq R} \{ \langle p, z \rangle - H(p) \} = h(z)$$

since h is convex and by Lemma 3.9. Hence, by definition, $V_\varepsilon \geq V$.

On the other hand, u_ε is the mild solution of

$$\lambda u_\varepsilon = \mathcal{L}u_\varepsilon + \langle F, u_{\varepsilon x} \rangle - H_\varepsilon(u_{\varepsilon x}) + \psi$$

and thus of

$$\lambda u_\varepsilon = \mathcal{L}u_\varepsilon + \langle F, u_{\varepsilon x} \rangle - H(u_{\varepsilon x}) + [\psi - H_\varepsilon(u_{\varepsilon x}) + H(u_{\varepsilon x})]$$

since $H - H_\varepsilon \in C_b(X)$ and $u_\varepsilon \in C_b^1(X)$. Then we can apply the comparison result obtained in section 3 and it yields

$$\|u_\varepsilon - u\| \leq \frac{1}{\lambda} \|H(u_{\varepsilon x}) - H_\varepsilon(u_{\varepsilon x})\| \leq \frac{\varepsilon R^2}{2\lambda}.$$

Moreover, we already know that $V \geq u$ and then we have

$$V \geq u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon = \lim_{\varepsilon \rightarrow 0} V_\varepsilon \geq V$$

and the proof is complete. \blacksquare

6. EXAMPLES

We shall work with the following

Hypothesis 6.1. Let X be a separable Hilbert space and let $\{e_k\}$ be a complete orthonormal system in X . We assume here that A and Q are of the following form:

$$Ae_k = -\alpha_k e_k, \quad Qe_k = \lambda_k e_k, \quad k \in \mathbb{N},$$

where $\{\alpha_k\}$ and $\{\lambda_k\}$ are sequences of positive numbers respectively increasing to $+\infty$ and decreasing to 0.

The following proposition is proved in [18].

PROPOSITION 6.2. Assume that Hypothesis 6.1 holds and that $\lambda_k = \alpha_k^{-r}$, $\forall k \in \mathbb{N}$ and for some $r \geq 0$. Then Hypotheses 2.2 and 2.5 are satisfied if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k^{1+r}} < +\infty \quad \text{and} \quad r < 1.$$

EXAMPLE 6.3. Let $C_N = [0, \pi]^N$ and $X = L^2(C_N)$, $N \leq 3$ and take the Laplace operator with Dirichlet conditions at the boundary defined as

$$D(A) = H^2(C_N) \cap H_0^1(C_N), \quad Ax = \Delta x, \quad \text{for } x \in D(A).$$

The operator A satisfies Hypothesis 2.1 and generates an analytic semi-group of compact operators. Moreover A satisfies Hypothesis 6.1 by taking, for $(n_1, \dots, n_N) \in \mathbb{N}^N$,

$$e_{n_1, \dots, n_N}(\xi) = \left(\frac{2}{\pi}\right)^{N/2} \sin n_1 \xi_1 \cdots \sin n_N \xi_N$$

and

$$\alpha_{n_1, \dots, n_N}(\xi) = n_1^2 + \cdots + n_N^2$$

so that, by ordering the eigenvalues, we obtain

$$\alpha_k \approx k^{2/N} \quad \text{as } k \rightarrow +\infty.$$

If we take, as in Proposition 6.2, $Qe_k = \alpha_k^{-r}$, $r \geq 0$, then Hypotheses 2.2 and 2.5 are fulfilled provided

$$\frac{N-2}{2} < r < 1$$

which is possible for $N \leq 3$. If $N = 1$ then we can take $r = 0$ which is the case studied in [4]. Define now

$$\psi(x) = \int_{C_N} \alpha(x(\xi)) d\xi, \quad h(z) = \frac{1}{2}|z| \quad F(x(\cdot))(\xi) = f(x(\xi))$$

where $\alpha \in C_b(\mathbb{R})$ and $f \in C_b(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$. Then $\psi \in C_b(X)$ and the hypotheses of Theorem 5.7 are satisfied. Therefore the results of Theorem 5.7 apply to the following optimal stochastic control problem. Minimize the Cost Functional

$$J(x; z) = \mathbb{E} \left\{ \int_0^{+\infty} e^{-\lambda s} \int_{C_N} [\alpha(y(s, \xi)) + \frac{1}{2}|z(s, \xi)|^2] d\xi ds \right\} \quad (6.1)$$

over all controls $z \in M_{\mathcal{W}}^2(t, T; X)$ satisfying $|z(s, \cdot)| \leq R$ almost surely for $s \in [0, +\infty[$ where the state $y(\cdot, \xi)$ is the mild solution of the differential stochastic equation

$$\begin{aligned} dy(s, \cdot) &= [A_\xi y(s, \xi) + f(x(s, \xi)) - z(s, \xi)] ds + \sqrt{Q} dW(s), \quad s > t \\ y(s, \xi) &= 0 \quad (s, \xi) \in [t, +\infty] \times \partial C_N \\ y(0, \xi) &= x(\xi), \quad \xi \in C_N \end{aligned} \quad (6.2)$$

driven by a White Noise W .

EXAMPLE 6.4. Let $X = L^2(C_N)$, and take the iterated Laplace operator with Dirichlet conditions at the boundary defined as

$$\begin{aligned} D(A_m) &= \{x \in H^{2m}(C_N), x, \Delta x, \dots, \Delta^{m-1}x = 0 \text{ on } \partial C_N\} \\ A_m x &= (-1)^{m-1} (\Delta)^m x, \quad \text{for } x \in D(A). \end{aligned}$$

The operator A_m (which occurs in elasticity theory when $m = 2$) satisfies Hypothesis 2.1-(i) and generates an analytic semigroup of compact operators. Moreover A_m satisfies Hypothesis 6.1 as in the case $m = 1$ but

$$\alpha_k \approx k^{(2m/N)} \quad \text{as } k \rightarrow +\infty.$$

So, if we take, as in Proposition 6.2, $Qe_k = \alpha_k^{-r}$, $r \geq 0$, then the hypotheses of Theorem 5.7 are fulfilled, provided

$$\frac{N-2m}{2m} < r < 1$$

which is possible for $N \leq 4m - 1$. If $N < 2m$ then we can take $r = 0$ (see [4]). In this case, Theorem 5.7 can be applied to the stochastic control problem (6.1) where, in the state equation (6.2), A is replaced by A_m .

REFERENCES

1. V. BARBU, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff, Groningen, 1976.
2. V. BARBU AND G. DA PRATO, "Hamilton–Jacobi Equations in Hilbert Spaces," Research Notes in Math., Pitman, Boston, 1982.
3. P. CANNARSA AND G. DA PRATO, Direct solution of a second order Hamilton–Jacobi equation in Hilbert Spaces, in "Stochastic Partial Differential Equations and Applications (G. Da Prato and L. Tubaro, Eds.), Research Notes in Mathematics, Vol. 268, pp. 72–85, Boston, 1992.
4. P. CANNARSA AND G. DA PRATO, Second order Hamilton–Jacobi equations in infinite dimensions, *SIAM J. Control Optim.* **29**, No. 2 (1991), 474–492.
5. P. CANNARSA AND G. DA PRATO, Infinite dimensional elliptic equations with Hölder-continuous coefficient, *Adv. Differential Equations* **1**, No. 3 (1996), 425–452.
6. S. CERRAI, A Hille–Yoshida Theorem for weakly continuous semigroups, *Semi-group Forum* **49** (1994), 349–367.
7. S. CERRAI AND F. GOZZI, Strong solutions of Cauchy problems associated to weakly continuous semigroups, *Differential Integral Equations* **8**, No. 3 (1995), 465–486.
8. A. CHOJNOWSKA-MICHALIK AND B. GOLDBY, Existence, uniqueness and invariant measures, for stochastic semilinear equations in Hilbert spaces, preprint.
9. P. L. CHOW AND J. L. MENALDI, Infinite dimensional Hamilton–Jacobi equations in Gauss–Sobolev spaces, preprint.
10. M. G. CRANDALL, H. ISHII, AND P. L. LIONS, User's guide to viscosity solutions of second order partial differential equations, *Bull. A.M.S. N. S.* **27**, No. 1 (1992), 1–67.
11. M. G. CRANDALL AND P. L. LIONS, Hamilton–Jacobi equations in infinite dimensions, Part I. Uniqueness of viscosity solutions, *J. Funct. Anal.* **62** (1985), 379–396. Part II. Existence of viscosity solutions, *J. Funct. Anal.* **65** (1986), 368–405. Part III, *J. Funct. Anal.* **68** (1986), 214–247.
12. G. DA PRATO, "Applications croissantes et équations d'évolution dans les espaces de Banach," *Institutiones Mathematicae*, Vol. II, Academic Press, London/New York, 1976.
13. G. DA PRATO, Some results on Bellmann equation in Hilbert spaces, *SIAM J. Contr. Optim.* **23** (1985), 61–71.
14. G. DA PRATO AND A. LUNARDI, On the Ornstein–Uhlenbeck operator in spaces of continuous functions, *J. Funct. Anal.* **131** (1995), 94–114.
15. G. DA PRATO AND J. ZABCZYK, Smoothing properties of transition semigroups in Hilbert spaces, *Stochast. Rep.* **35** (1991), 63–77.
16. G. DA PRATO AND J. ZABCZYK, "Stochastic Equations in Infinite Dimensions," *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, U.K., 1992.
17. W. H. FLEMING AND H. M. SONER, "Controlled Markov Processes and Viscosity Solutions," Springer-Verlag, Berlin/New York, 1993.
18. F. GOZZI, Regularity of solutions of a second order Hamilton–Jacobi equation and application to a control problem, *Preprints di Matematica* n. 9, 1994, Scuola Normale Superiore, Pisa, *Comm. Partial Differential Equations* **20**, No. 5 & 6 (1995), 775–826.

19. F. GOZZI, Global existence of solutions for a second order Hamilton–Jacobi equation in Hilbert spaces, preprint n.2.192.852, Marzo 1995, Dipartimento di Matematica, Università di Pisa, *J. Math. Anal. Appl.* **198** (1996), 399–443.
20. T. HAVARNEANU, Existence for the dynamic programming equation of control diffusion processes in Hilbert space, *Nonlinear Anal. Theory Meth. Appl.* **9** (1985), 619–629.
21. L. HÖRMANDER, Hypoelliptic second order differential equations, *Acta Math.* **119** (1967), 147–171.
22. J. M. LASRY AND P. L. LIONS, A remark on regularization in Hilbert spaces, *Israel J. Math.* **55** (1986), 147–171.
23. P. L. LIONS, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part I. The case of bounded stochastic evolution, *Acta Math.* **161** (1988), 243–278. Part II. “Optimal control of Zakai’s equation,” Lecture Notes in Mathematics, Vol. 1390, pp. 147–170, (G. Da Prato and L. Tubaro, Eds.), Springer-Verlag, Berlin, 1989. Part III: Uniqueness of viscosity solutions for general second order equations, *J. Funct. Anal.* **86** (1989), 1–18.
24. A. LUNARDI, “Analytic Semigroups and Optimal Regularity in Parabolic Problems,” Birkhäuser, Basel, 1995.
25. A. S. NEMIROVSKI AND S. M. SEMENOV, The polynomial approximation of functions in Hilbert spaces, *Mat. Sb. N.S.* **92** (1973), 134, 257–281.
26. S. POLIDORO, “Su una classe di operatori ultraparabolici di tipo Kolmogorev–Fokker–Planck” Tesi di Dottorato, Università di Bologna, 1994.
27. T. SEIDMAN, How violent are fast controls? *Math. Control Sign. Syst.* **1** (1988), 89–95.
28. A. SWIECH, “Viscosity Solutions of Fully Nonlinear Partial Differential Equations with “Unbounded” Terms in Infinite Dimensions,” Ph.D. Thesis, University of California, Santa Barbara, July 1993.